Math 299 **Lecture 20: Sections 8.1 - 8.3**
Equivalence Relations.

Definition 1 (*A relation*): Let X and Y be sets.

A **relation** R **from** X **to** Y is a subset of $X \times Y$.

Example 1. Let $A = \{1, 2\}$, $B = \{a, b, c\}$. List 3 relations from A to B. How many possible relations are there from A to B?

Definition 2. Let R be a relation from X to Y.

The **domain of** R is

$$dom(R) = \{x \in X : (x, y) \in R \text{ for some } y \in Y\}.$$

The **range of** R is

$$range(R) = \{y \in Y : (x, y) \in R \text{ for some } x \in X\}.$$

Notation: If $(x, y) \in R$, we write xRy.

Definition 3 (*Inverse Relation*): Let R be a relation from X to Y. The **inverse relation of** R is

$$R^{-1} = \{(y, x) : (x, y) \in R\}.$$

Example 2. Find R^{-1} for the relations you constructed above.

- What is the domain of R^{-1}?

- What is the range of R^{-1}?

Properties of Relations

Definition 4. A relation from a set X to X is **reflexive** if $xRx \, \forall x \in X$.

Give an example of a relation which is reflexive and a relation which is not reflexive.

Examples. Determine if the following relations are reflexive or not.

(1) The relation S is defined on $\mathbb{R}$ by aSb if $a < b$.

(2) The relation $\sim$ is defined on $\mathbb{R}$ by $x \sim y$ if $x \mid y$.

(3) The relation Q is defined on $\mathbb{R}$ by $(a, b) \in Q$ if $a \leq b$.

Definition 5. A relation from a set X to X is **symmetric** if whenever xRy, then yRx.

Give an example of a relation which is symmetric and a relation which is not symmetric.

Examples. Determine if the following relations are symmetric or not.

(1) The relation S is defined on $\mathbb{R}$ by aSb if $a < b$.

(2) The relation $\sim$ is defined on $\mathbb{R}$ by $x \sim y$ if $x \mid y$.

(3) The relation P is defined on $\mathbb{R}$ by $(a, b) \in P$ if $\frac{a}{b} \in \mathbb{Q}$.

Definition 6. A relation from a set X to X is **transitive** if whenever xRy and yRz, then xRz.

Give an example of a relation which is transitive and a relation which is not transitive.

Examples. Determine if the following relations are transitive or not.

(1) The relation S is defined on $\mathbb{R}$ by aSb if $a < b$.

(2) The relation $\sim$ is defined on $\mathbb{R}$ by $x \sim y$ if $x \mid y$.

(3) The relation M is defined on $\mathbb{R}$ by $(a, b) \in Q$ if $|a - b| < 1$.

Equivalence Relations

Definition 7 (*An equivalence relation*): A relation from a set X to X is an **equivalence relation** if it is reflexive, symmetric and transitive.

Definition 8. For an equivalence relation R defined on a set X and for $a \in X$, the set

$$[a] = \{x \in \ X : xRa\}$$

consisting of all elements x related to a is called the **equivalence class** of a.

- Can $[a]$ be an empty set?

- Find the equivalence classes determined by

$$R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$$

Examples

(a) Are the following relations equivalence relations?

(b) If not, explain why not.

(c) If they are, prove that they are and

(d) determine the distinct equivalence classes.

1. The relation $\simeq$ is defined on the set $H = \{5k : k \in \mathbb{Z}\}$ by $a \simeq b$ if $a - b \in H$.

2. The relation P is defined on $\mathbb{R} \times \mathbb{R}$ by $(a, b)R(c, d)$ if $|a - c| \leq 3$.

3. The relation $\sim$ is defined on $\mathbb{R} \times \mathbb{R}$ by $(a, b) \sim (c, d)$ if $a^2 - b = c^2 - d$.